DISTRIBUTIONAL SOLUTIONS OF THE STATIONARY NONLINEAR SCHRÖDINGER EQUATION: SINGULARITIES, REGULARITY AND EXPONENTIAL DECAY

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ABSTRACT. We consider the nonlinear Schrödinger equation $-\Delta u + V(x)u = \Gamma(x)|u|^{p-1}u$ in $\mathbb{R}^n$ where the spectrum of $-\Delta + V(x)$ is positive. In the case $n \geq 3$ we use variational methods to prove that for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ there exist distributional solutions with a point singularity at the origin provided $\varepsilon > 0$ is sufficiently small and V, Γ are bounded on $\mathbb{R}^n \setminus B_1(0)$ and satisfy suitable Hölder-type conditions at the origin. In the case $n = 1, 2$ or $n \geq 3, 1 < p < \frac{n}{n-2}$, however, we show that every distributional solution of the more general equation $-\Delta u + V(x)u = g(x, u)$ is a bounded strong solution if V is bounded and g satisfies certain growth conditions.

1. Introduction and main result

In this paper we investigate distributional solutions of the stationary nonlinear Schrödinger equation (NLS)

$$-\Delta u + V(x)u = \Gamma(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.1}$$

for $n \in \mathbb{N}$ and $1 < p < \frac{n+2}{(n-2)_+}$. The NLS (1.1) has been receiving much attention due to its applicability in different fields of mathematical physics, e.g. nonlinear optics, mean field theory, Bose-Einstein condensates. Spatially localized soliton-like solutions $u \in H^1(\mathbb{R}^n)$ of (1.1) can be expected whenever 0 does not belong to the spectrum of $-\Delta + V(x)$. Ever since pioneering work of Strauss [25], Berestycki-Lions [1,2], Stuart [27] a lot of results on existence and non-existence of ground states/bound states, multiplicity, asymptotic behaviour, bifurcation phenomena etc. have been obtained. In the case where V, Γ are positive constants the results of Gidas, Ni, Nirenberg [5] and Li [13] apply and show that all positive solutions decaying to 0 at infinity must be radially symmetric. Recently, due to new developements in photonic crystals, the case of periodic coefficients V, Γ has been studied, cf. Pankov [19] and Szulkin-Weth [28]. In all of these works the solutions were weak (or classical) solutions belonging to $H^1(\mathbb{R}^n)$.

More recently, distributional solutions of nonlinear elliptic boundary value problems like (1.1) have been studied. In the context of bounded domains various classes of *very weak solutions*, i.e. subclasses of distributional solutions with prescribed Dirichlet boundary data, have

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been investigated, cf. Stampacchia [23], Brézis et al. [3], Quittner-Souplet [21], McKenna-Reichel [14], McKenna et al. [10], del Pino et al. [4]. In the context of the Yamabe problem, Pacard [17,18] and Mazzeo-Pacard [16] have also studied distributional solutions of nonlinear boundary value problems similar to (1.1). In many of the above mentioned results the following phenomenon occurs: for a range of exponents $1 < p < p^*$ all very weak solutions turn out to have no singularities and are indeed bounded weak/classical solutions of the nonlinear elliptic problem, whereas for $p^* < p < p^* + \varepsilon$ unbounded very weak solutions were shown to exist.

In the present paper we show a similar phenomenon for the NLS (1.1). The singular distributional solutions that we find have some properties in common with $H^1(\mathbb{R}^n)$-solutions of (1.1), e.g. they decay exponentially fast at infinity. On the other hand, even in cases where there are no non-trivial $H^1(\mathbb{R}^n)$ solutions, singular distributional solutions can be shown to exist, cf. Remark 4. Let us point out two further interesting aspects of singular distributional solutions of (1.1): First, if V, Γ satisfy the conditions given below and are radially symmetric such that Γ is positive and radially decreasing and V is positive and radially increasing then by Li's result, cf. [13], all weak/classical non-negative solutions which decay to 0 at infinity must be radially symmetric. However, using Theorem 2 one can construct a distributional solution which is not radially symmetric having a single point singularity at the origin although V, Γ are radially symmetric with respect to some point $x_0 \in \mathbb{R}^n \setminus \{0\}$. Second, let us view singular distributional solutions from the point of view of numerical approximations. From the outcome of one numerical calculation of an approximate solution to (1.1) it is impossible to tell if the computed result approximates a singular disitributional solution or a very large weak/classical solution. Mesh refinements may help to clarify it. However, from our Theorem 3 it is clear that below the exponent $p^* = \frac{n}{n-2}$ (which is smaller than the usual critical exponent $\frac{n+2}{n-2}$) no such singular distributional solutions can exist.

Our tools range from linear Schrödinger theory, calculus of variations, Green's functions to the use of singular integral estimates. Results concerning exponential decay of eigenfunctions are proved by an adapted version of Agmon's method (cf. [9], [11], [12]).

In our first result Theorem 2 we follow the ideas of [10], [16] to prove the existence of an unbounded exponentially decaying distributional solution of (1.1) when $n \geq 3$ and $\frac{n}{n-2} < p < \frac{n}{n-2} + \varepsilon$ for $\varepsilon > 0$ sufficiently small. We concentrate on the construction of distributional solutions with one point singularity at the origin. To this end we assume the following conditions on $V, \Gamma : \mathbb{R}^n \to \mathbb{R}$:

(H1) $V \in L^\infty\big(\mathbb{R}^n \setminus B_1(0)\big)$ and there are constants $C_1 > 0$ and $\alpha \geq \frac{n-6}{2}$ such that

$$|V(x)| \leq C_1 |x|^\alpha \quad \text{for almost all } x \in B_1(0).$$

(H2) $\Sigma := \min \sigma(-\Delta + V(x)) > 0$ where σ denotes the L^2-spectrum.

(H3) $\Gamma \in L^\infty(\mathbb{R}^n)$ and there are constants $C_2 > 0$ and $\beta > \frac{n-2}{2}$ such that

$$|\Gamma(x) - \Gamma(0)| \leq C_2 |x|^\beta \quad \text{for almost all } x \in B_1(0),$$

where $\Gamma(0) > 0$. Rescaling (1.1) we can assume w.l.o.g. $\Gamma(0) = 1$.

In our second result Theorem 3 we show that for $1 < p < \frac{n}{(n-2)_+}$ the equation

$$-\Delta u + V(x)u = g(x,u) \quad \text{in } \mathbb{R}^n \tag{1.2}$$

and in particular (1.1) does not admit positive locally unbounded distributional solutions provided $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies

$$-C_3 + C_4 s^p \le g(x,s) \le C_3 + C_5 s^p \qquad (x \in \mathbb{R}^n, s \ge 0). \tag{1.3}$$

where $C_3, C_4, C_5 > 0$. We also obtain a global boundedness and a global regularity result in the case g satisfies

$$|g(x,s)| \le C_6 \left(|s| + |s|^p\right) \qquad (x \in \mathbb{R}^n, s \in \mathbb{R}), \tag{1.4}$$

where $C_6 > 0$. In addition we find that distributional solutions of (1.2) decay exponentially in the case

$$\lim_{s\to 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{|g(x,s)|}{|s|} = 0. \tag{1.5}$$

If remains open if or if not unbounded distributional solutions exists in the borderline case $p = \frac{n}{n-2}$.

All our results are built on the following notion of a distributional solution.

Definition 1. *Let $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $|g(x,s)| \le C(1 + |s|^p)$ for all $s \in \mathbb{R}$, almost all $x \in \mathbb{R}^n$ and some $C > 0$, $1 < p < \infty$. A function $u \in L^p_{loc}(\mathbb{R}^n)$ with $Vu \in L^1_{loc}(\mathbb{R}^n)$ is called a distributional solution of* (1.2) *if*

$$\int_{\mathbb{R}^n} u(-\Delta\varphi + V(x)\varphi)\,dx = \int_{\mathbb{R}^n} g(x,u)\varphi\,dx \quad \textit{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

In contrast, a function $u \in L^p_{loc}(\mathbb{R}^n)$ with $\nabla u, Vu \in L^1_{loc}(\mathbb{R}^n)$ is called a *weak solution* of (1.2) if

$$\int_{\mathbb{R}^n} \nabla u \nabla\varphi + V(x)u\varphi\,dx = \int_{\Omega} g(x,u)\varphi\,dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{1.6}$$

Similarly, we say that u is a *weak solution of* (1.2) *on an open subset* $\Omega \subset \mathbb{R}^n$ if (1.6) holds for all $\varphi \in C_c^\infty(\Omega)$. A function $u \in L^p_{loc}(\mathbb{R}^n)$ with $-\Delta u, Vu \in L^1_{loc}(\mathbb{R}^n)$ will be called a *strong solution* of (1.2) if $-\Delta u + Vu = g(x,u)$ holds almost everywhere in $\mathbb{R}^n$.

Our main results are the following two theorems.

Theorem 2 (Supercritical case)**.** *Let the assumptions (H1),(H2),(H3) hold and let $n \ge 3$. Then there exists $\varepsilon > 0$ such that for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ there is a distributional solution U of* (1.1) *with the following properties:*

(i) $\operatorname{ess\,sup}_{B_\delta(0)} U = +\infty$ *for all $\delta > 0$ and $U \in L^q(\mathbb{R}^n)$ for all $1 \le q < \frac{n(p-1)}{2}$.*
(ii) *For all $\delta > 0$ the function $U \in H^1(\mathbb{R}^n \setminus B_\delta)$ is a weak solution of* (1.1) *on $\mathbb{R}^n \setminus B_\delta$.*
(iii) *For all $\mu \in (0, \sqrt{\Sigma})$ there is $C_\mu > 0$ such that $|U(x)| \le C_\mu e^{-\mu|x|}$ if $|x| \ge 1$.*
(iv) *If in addition $\Gamma \ge 0$ then U can be chosen to satisfy $U \ge 0$.*

Theorem 3 (Subcritical case). *Let $n \in \mathbb{N}$, $1 < p < \frac{n}{(n-2)_+}$, $V \in L^\infty(\mathbb{R}^n)$, let $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let u be a distributional solution of* (1.2)*.*

(1) (Local regularity) *If g satisfies* (1.3) *and if $u \geq 0$ then $u \in W^{1,\infty}_{loc}(\mathbb{R}^n) \cap W^{2,q}_{loc}(\mathbb{R}^n)$ for all $q \in [1, \infty)$.*
(2) (Global regularity) *If g satisfies* (1.4) *and if $u \in L^p(\mathbb{R}^n)$ then $u \in W^{1,q}(\mathbb{R}^n) \cap W^{2,q'}(\mathbb{R}^n)$ for all $q \in [p, \infty]$, $q' \in [p, \infty)$. If in addition V satisfies (H2) and g satisfies* (1.5) *then $u \in W^{1,q}(\mathbb{R}^n) \cap W^{2,q'}(\mathbb{R}^n)$ for all $q \in [1, \infty]$, $q' \in (1, \infty)$ and for all $0 < \mu < \sqrt{\Sigma}$ there is $C_\mu > 0$ such that $|u(x)| \leq C_\mu e^{-\mu|x|}$ in $\mathbb{R}^n$.*

In both cases u is a strong solution of (1.2)*.*

Remark 4.

(1) *Note that for every compact set $K \subset \mathbb{R}^n$ the potential $V = 1_{\mathbb{R}^n \setminus K}$ satisfies (H1),(H2) for every $\alpha \geq \frac{n-6}{2}$.*
(2) *In the case $n = 3, 4, 5, 6$ Theorem 2 applies to every measurable function V which satisfies $0 < V_0 \leq V \leq V_1$ almost everywhere for some positive constants V_0, V_1. For instance we find an unbounded distributional solution of the equation $-\Delta u + V(x)u = |u|^{p-1}u$ in the case $V \in W^{1,\infty}(\mathbb{R}^n)$ is strictly monotone in some direction $v \in \mathbb{R}^n$, e.g. $V(x) = \pi + \arctan(xv)$. This is quite interesting given the fact that in this case the only $H^1(\mathbb{R}^n)$-solution is the trivial one. Indeed, if $u \in H^1(\mathbb{R}^n)$ is a solution then $u \in H^2(\mathbb{R}^n)$ (see Theorem 3,(2)) and testing the equation with $\partial_v u$ leads to*

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \Big(\nabla u \nabla(\partial_v u) + V u \partial_v u - |u|^{p-1} u \partial_v u\Big)\, dx \\ &= \int_{\mathbb{R}^n} \partial_v \Big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\Big) + \frac{1}{2}\int_{\mathbb{R}^n} V \partial_v(|u|^2)\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^n} (\partial_v V)|u|^2\, dx \end{aligned}$$

by density of $C_0^\infty(\mathbb{R}^n)$ in $H^2(\mathbb{R}^n)$. Hence, $u \equiv 0$ because $\partial_v V < 0$ in $\mathbb{R}^n$. The above result is due to Tanaka [29], see also Theorem 1.3. in [15].
(3) *If we add regularity assumptions on V and g in Theorem 3 then elliptic regularity theory will give better results. If V and g are both C^∞-functions, say, then every positive distributional solution u of* (1.2) *is in fact a classical solution. Similarly, if in Theorem 2 V, g are both C^∞-functions then part (ii) of Theorem 2 gives $U \in C^\infty(\mathbb{R}^n \setminus \{0\})$.*
(4) *By a suitable choice of test functions one can extend the local regularity result of Theorem 3 to possibly sign-changing solutions of equation* (1.2) *where the nonlinearity satisfies the more general inequality $|g(x, u)| \leq c(1 + |u|^p)$.*

In the proof of Theorem 2 we always require $0 < \varepsilon < \frac{2}{n-2}$ so that $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and variational methods are applicable. Estimates involving $p - \frac{n}{n-2}$ will be carried out explicitly. Throughout the paper $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ is the open ball of radius r in $\mathbb{R}^n$ and c is a constant which can change from line to line but which is independent of p. We use the symbol

$\frac{n}{(n-2)_+}$ to denote the value ∞ for $n=1,2$ and the value $\frac{n}{n-2}$ in the case $n \geq 3$. Similarly the symbols $\frac{n}{(n-1)_+}$, $\frac{2n}{(6-n)_+}$ etc. are used. The assumptions (H1), (H2) imply that the bilinear form

$$\langle u,v\rangle_V := \int_{\mathbb{R}^n} \big(\nabla u \nabla v + V(x)uv\big)\,dx \qquad (u,v\in H^1(\mathbb{R}^n)) \tag{1.7}$$

generates a norm $\|\cdot\|_V$ on $H^1(\mathbb{R}^n)$ which is equivalent to the standard H^1-norm $\|\cdot\|$.

Finally let us recall the definition of the Kato class K_n, cf. [22]. Let $h_n(x,y) = |x-y|^{2-n}$ for $n \geq 3$, $h_2(x,y) = -\log|x-y|$ and $h_1(x,y) = 1$. A measurable function $W : \mathbb{R}^n \to \mathbb{R}$ belongs to K_n, $n \in \mathbb{N}$ if

$$\lim_{\rho\to 0} \sup_{x\in\mathbb{R}^n} \int_{\{|x-y|\leq\rho\}} h_n(x,y)|W(y)|\,dy = 0\ , \quad n \geq 2,$$

$$\sup_{x\in\mathbb{R}^n} \int_{\{|x-y|\leq 1\}} |W(y)|\,dy < \infty, \quad n = 1.$$

A norm on K_n is given by (cf. [22], p.453, (A15))

$$\|W\|_{K_n} := \sup_{x\in\mathbb{R}^n} \int_{\{|x-y|\leq 1\}} h_n(x-y)|W(y)|\,dy.$$

If $\Omega \subset \mathbb{R}^n$ is open we denote by $K_n(\Omega)$ the set of measurable functions $W : \mathbb{R}^n \to \mathbb{R}$ such that $W1_\Omega$ lies in the Kato class K_n. The mapping $\|W\|_{K_n(\Omega)} := \|W1_\Omega\|_{K_n}$ defines a seminorm on $K_n(\Omega)$. For every $q \in (\frac{n}{2}, \infty]$ there exists a constant $c_q > 0$ such that

$$\|W\|_{K_n(\Omega)} \leq c_q \sup_{y\in\Omega} \|W\|_{L^q(B_1(y))} \tag{1.8}$$

whenever the right hand side is finite.

2. Proof of Theorem 2

Our existence proof of an unbounded distributional solution U is inspired by [10], [16]. We start by constructing an approximate solution u_0 of equation (1.1) which is unbounded near 0. Then we determine a functional $J : H^1(\mathbb{R}^n) \to \mathbb{R}$ such that every critical point $\tilde u \in H^1(\mathbb{R}^n)$ of J gives rise to a distributional solution $U := u_0 + \tilde u$ of (1.1) which has the desired properties. The main difficulty will be to prove that J has a critical point. The proof of the parts (i) and (ii), (iii), (iv) will be given in section 2.4, 2.5, 2.6 respectively.

2.1. **Construction of an unbounded approximate solution.** For $p > \frac{n}{n-2}$ let the function $u_1 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be defined by

$$u_1(x) := c_{n,p}|x|^{-\frac{2}{p-1}} \quad \text{where} \quad c_{n,p} = \Big(\frac{2}{p-1}(n-2-\frac{2}{p-1})\Big)^{\frac{1}{p-1}}. \tag{2.1}$$

Notice that $c_{n,p} \to 0$ as $p \searrow \frac{n}{n-2}$ and

$$-\Delta u_1 = u_1^p \qquad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

Replacing u_1 outside a suitable ball B_ρ by an exponentially decreasing classical solution u_2 of

$$-\Delta u_2 + u_2 = u_2^p \qquad \text{in } \mathbb{R}^n \setminus B_\rho \tag{2.3}$$

we define the approximate solution

$$u_0(x) := \begin{cases} u_1(x), & x \in B_\rho, \\ u_2(x), & x \in \mathbb{R}^n \setminus B_\rho. \end{cases} \tag{2.4}$$

It turns out that such a function u_0 can be constructed with properties stated next. To state the Proposition let us define

$$\partial_\nu^+ u_0(x) = \lim_{t\to 0+} \frac{u_0(x) - u_0(x - t\nu(x))}{t}, \qquad \partial_\nu^- u_0(x) = \lim_{t\to 0+} \frac{u_0(x + t\nu(x)) - u_0(x)}{t}$$

for $\nu(x) = \frac{x}{|x|}$ whenever the limits exist.

Proposition 5 (Existence of an approximate solution)**.** *Let $n \in \mathbb{N}, n \geq 3$. Then there exists a radius $\rho \geq 1$ and a constant $c > 0$ such that for all $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$ there is a radially symmetric function $u_0 : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ with the following properties:*

(i) *$u_0 \in C^2(B_\rho \setminus \{0\})$ solves (2.2) in $B_\rho \setminus \{0\}$ in the classical sense.*
(ii) *$u_0 \in C^2(\mathbb{R}^n \setminus \overline{B}_\rho)$ solves (2.3) in $\mathbb{R}^n \setminus \overline{B}_\rho$ in the classical sense.*
(iii) *$u_0 \in C(\mathbb{R}^n \setminus \{0\})$ and all first and second order derivatives of u_0 admit continuous extensions to ∂B_ρ from either side. Moreover, for all $\delta > 0$ we have $u_0 \in H^1(\mathbb{R}^n \setminus B_\delta)$.*
(iv) *$\lim_{x\to 0} u_0(x) = +\infty$.*
(v) *$|\partial_\nu^+ u_0(x) - \partial_\nu^- u_0(x)| \leq c\, c_{n,p}$ for all $x \in \partial B_\rho$.*
(vi) *u_0 satisfies the estimate*

$$u_0(x) \leq \begin{cases} c_{n,p}|x|^{-\frac{2}{p-1}} & \text{for } x \in B_\rho, \\ c_{n,p} e^{-\frac{|x|-\rho}{2}} & \text{for } x \in \mathbb{R}^n \setminus B_\rho. \end{cases} \tag{2.5}$$

In particular, $u_0 \in L^q(\mathbb{R}^n)$ for all $q \in [1, \frac{n(p-1)}{2})$.

For a proof of this result we refer to Appendix A.

2.2. **Variational setting.** Given u_0 from Proposition 5 we prove existence of an unbounded distributional solution U of (1.1) using the ansatz

$$U := u_0 + \tilde{u} \tag{2.6}$$

where $\tilde{u} \in H^1(\mathbb{R}^n)$ will be constructed as a local minimizer of a suitable functional $J : H^1(\mathbb{R}^n) \to \mathbb{R}$. Once the existence of $\tilde{u}$ is shown we will see that $U := u_0 + \tilde{u}$ is a weak solution of (1.1) on $\mathbb{R}^n \setminus B_\delta$ for every $\delta > 0$ and a distributional solution of (1.1) on $\mathbb{R}^n$. The definition of J stems from the following motivation.

For a fixed test function $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have by Proposition 5

$$\int_{\mathbb{R}^n} (\nabla u_0 \nabla \varphi + V(x) u_0 \varphi)\, dx = \int_{\mathbb{R}^n} u_0^p \varphi\, dx + \oint_{\partial B_\rho} (\partial_\nu^+ u_0 - \partial_\nu^- u_0)\varphi\, d\sigma \tag{2.7}$$

$$+\int_{B_\rho} V(x)u_0\varphi\,dx + \int_{\mathbb{R}^n\setminus B_\rho}(V(x)-1)u_0\varphi\,dx$$

Since we want U to be a weak solution of (1.1) in $\mathbb{R}^n \setminus B_\delta$ for all $\delta > 0$ we require

$$\int_{\mathbb{R}^n}(\nabla U\nabla\varphi + V(x)U\varphi)\,dx = \int_{\mathbb{R}^n}\Gamma(x)|U|^{p-1}U\varphi\,dx.$$

Hence, the function $\tilde{u} \in H^1(\mathbb{R}^n)$ that we seek must satisfy

$$\begin{aligned}\int_{\mathbb{R}^n}(\nabla\tilde{u}\nabla\varphi + V(x)\tilde{u}\varphi)\,dx &= \int_{\mathbb{R}^n}\Big(\Gamma(x)|u_0+\tilde{u}|^{p-1}(u_0+\tilde{u}) - u_0^p\Big)\varphi\,dx\\ &\quad - \int_{B_\rho}V(x)u_0\varphi\,dx - \int_{\mathbb{R}^n\setminus B_\rho}(V(x)-1)u_0\varphi\,dx\\ &\quad - \oint_{\partial B_\rho}(\partial_\nu^+u_0 - \partial_\nu^-u_0)\varphi\,d\sigma.\end{aligned} \tag{2.8}$$

Thus, we will look for critical points $\tilde{u} \in H^1(\mathbb{R}^n)$ of the functional $J : H^1(\mathbb{R}^n) \to \mathbb{R}$ given by

$$J[u] := \frac{1}{2}\|u\|_V^2 - J_1[u] - J_2[u] + J_3[u] \tag{2.9}$$

where $\|\cdot\|_V$ is defined by (1.7) and $J_i : H^1(\mathbb{R}^n) \to \mathbb{R}$ $(i = 1, 2, 3)$ are defined by

$$\begin{aligned}J_i[u] &= \int_{\mathbb{R}^n}F_i(u,x)\,dx, \quad i = 1, 2,\\ J_3[u] &= \int_{B_\rho}V(x)u_0u\,dx + \int_{\mathbb{R}^n\setminus B_\rho}(V(x)-1)u_0u\,dx + \oint_{\partial B_\rho}(\partial_\nu^+u_0 - \partial_\nu^-u_0)\gamma(u)\,d\sigma.\end{aligned}$$

Here $\gamma : H^1(\mathbb{R}^n) \to L^2(\partial B_\rho)$ denotes the trace operator and the functions $F_1, F_2 : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are given by

$$\begin{aligned}F_1(s,x) &= \frac{1}{p+1}\big(|s+u_0(x)|^{p+1} - u_0(x)^{p+1} - (p+1)u_0(x)^p s\big),\\ F_2(s,x) &= \frac{\Gamma(x)-1}{p+1}\big(|u_0(x)+s|^{p+1} - u_0(x)^{p+1}\big).\end{aligned}$$

We will prove in Proposition 6 that J is well-defined and continuously Fréchet-differentiable.

In order to find a positive distributional solution of (1.1) in the case $\Gamma \geq 0$ we introduce the functional $\hat{J} : H^1(\mathbb{R}^n) \to \mathbb{R}$ given by

$$\hat{J}[u] := \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^n}\hat{F}_1(u,x)\,dx - \int_{\mathbb{R}^n}\hat{F}_2(u,x)\,dx + J_3[u] \tag{2.10}$$

where

$$\begin{aligned}\hat{F}_1(s,x) &= \frac{1}{p+1}\big((s+u_0(x))_+^{p+1} - u_0(x)^{p+1} - (p+1)u_0(x)^p s\big),\\ \hat{F}_2(s,x) &= \frac{\Gamma(x)-1}{p+1}\big((u_0(x)+s)_+^{p+1} - u_0(x)^{p+1}\big)\end{aligned}$$

The results of the upcoming section will hold for both J and $\hat{J}$ due to the fact that the inequalities (2.11),(2.12),(2.19),(2.20) and thus (2.13)-(2.16),(2.21) also hold for $\hat{F}_1, \hat{F}_2$.

2.3. **Existence of a critical point.** The proof of Theorem 2 relies on the following results. First we show in Proposition 6 that the functional J is well-defined and continuously Fréchet-differentiable for all $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$. In Proposition 8 we prove next $J[u] \geq m > 0$ for all $u \in H^1(\mathbb{R}^n)$ with $\|u\| = r_0$ and all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ for appropriately chosen $m, r_0, \varepsilon > 0$. Using Ekeland's variational principle we then prove in Proposition 9 the existence of a critical point $\tilde{u}$ of J. Finally, in Lemma 10 we show that $U := u_0 + \tilde{u}$ indeed defines an unbounded distributional solution of (1.1).

We start by proving that J is well-defined and continuously Fréchet-differentiable.

Proposition 6. *Let the assumptions of Theorem 2 hold. Then the functional J given by* (2.9) *is well-defined and continuously Fréchet-differentiable for all $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$ with Fréchet-derivative*

$$J'[u](\varphi) = \langle u, \varphi \rangle_V - \int_{\mathbb{R}^n} \big(F_1'(u,x)\varphi + F_2'(u,x)\varphi\big)\,dx + J_3[\varphi].$$

Here $'$ refers to the partial derivative with respect to the first variable.

Proof. J is well-defined: First we show that J_1, J_2 are well-defined. The estimates

$$|F_1(s,x)| \leq c\big(u_0(x)^{p-1}s^2 + |s|^{p+1}\big), \tag{2.11}$$

$$|F_2(s,x)| \leq c|\Gamma(x) - 1|\,\big(u_0(x)^p|s| + |s|^{p+1}\big) \tag{2.12}$$

together with (2.5) and (H3) imply

$$|F_1(s,x)| \leq c \begin{cases} |s|^{p+1} + c_{n,p}^{p-1}\frac{|s|^2}{|x|^2}, & \text{if } x \in B_\rho \\ |s|^{p+1} + c_{n,p}^{p-1}|s|^2, & \text{if } x \in \mathbb{R}^n \setminus B_\rho, \end{cases} \tag{2.13}$$

$$|F_2(s,x)| \leq c \begin{cases} |s|^{p+1} + c_{n,p}^{p}|x|^{\beta - \frac{p+1}{p-1}}\frac{|s|}{|x|}, & \text{if } x \in B_\rho \\ |s|^{p+1} + c_{n,p}^{p}e^{-\frac{p}{2}|x-\rho|}|s|, & \text{if } x \in \mathbb{R}^n \setminus B_\rho. \end{cases} \tag{2.14}$$

By Hardy's inequality we obtain from (2.13)

$$\begin{aligned} |J_1[u]| &\leq c\left(\int_{\mathbb{R}^n} |u|^{p+1}\,dx + c_{n,p}^{p-1}\int_{B_\rho} \frac{|u|^2}{|x|^2}\,dx + c_{n,p}^{p-1}\int_{\mathbb{R}^n\setminus B_\rho} u^2\,dx\right) \\ &\leq c(\|u\|^{p+1} + c_{n,p}^{p-1}\|u\|^2). \end{aligned} \tag{2.15}$$

Since $\beta > \frac{n-2}{2}$ by (H3) and $p > \frac{n}{n-2}$ we have $\||x|^{\beta - \frac{p+1}{p-1}}\|_{L^2(B_\rho)} \leq c$. Hence (2.14) and Hardy's inequality imply

$$\begin{aligned} |J_2[u]| &\leq c\left(\int_{\mathbb{R}^n} |u|^{p+1}\,dx + c_{n,p}^{p}\int_{B_\rho} |x|^{\beta - \frac{p+1}{p-1}}\frac{|u|}{|x|}\,dx + c_{n,p}^{p}\int_{\mathbb{R}^n\setminus B_\rho} e^{-\frac{p}{2}|x-\rho|}|u|\,dx\right) \\ &\leq c(\|u\|^{p+1} + c_{n,p}^{p}\|u\|). \end{aligned} \tag{2.16}$$

Therefore J_1, J_2 are well-defined.

It remains to prove that J_3 is well-defined. From $\alpha \geq \frac{n-6}{2}$ by assumption (H1) and $p > \frac{n}{n-2}$ we infer $|x|^{\alpha+\frac{p-3}{p-1}} \in L^2(B_\rho)$ with

$$D(p) := \||x|^{\alpha+\frac{p-3}{p-1}}\|_{L^2(B_\rho)} \leq c\left(2\alpha + \frac{2p-6}{p-1} + n\right)^{-1/2}. \tag{2.17}$$

Therefore (2.5) and Hardy's inequality yield

$$\int_{B_\rho} |V(x)u_0 u|\,dx \leq c\,c_{n,p}\int_{B_\rho} |x|^{\alpha+\frac{p-3}{p-1}}\frac{|u|}{|x|}\,dx \leq c\,c_{n,p}D(p)\|u\| \tag{2.18}$$

so that the first integral in J_3 is well-defined on $H^1(\mathbb{R}^n)$. The remaining two integrals in J_3 are also well-defined on $H^1(\mathbb{R}^n)$ since u_0 decays exponentially at infinity and since the one-sided derivatives in the boundary integral exist by Proposition 5 (i),(ii). Hence, J is well-defined.

Fréchet-differentiability: Since J_3 is linear we only have to deal with J_1, J_2. Similar to the calculations above we get for $i = 1, 2$, $x \in \mathbb{R}^n$, $s, t \in \mathbb{R}$

$$\begin{aligned}
&|F_i(s+t,x) - F_i(s,x) - tF_i'(s,x)| \\
&\leq c\big||u_0(x)+s+t|^{p+1} - |u_0(x)+s|^{p+1} - (p+1)|u_0(x)+s|^{p-1}(u_0(x)+s)t\big| \\
&\leq c\left(|u_0(x)+s|^{p-1}t^2 + |t|^{p+1}\right) \\
&\leq c\left(u_0(x)^{p-1}t^2 + |s|^{p-1}t^2 + |t|^{p+1}\right),
\end{aligned} \tag{2.19}$$

where for $i = 2$ we estimated $|\Gamma(x) - 1| \leq \|\Gamma\|_\infty + 1$. Hardy's and Sobolev's inequality and the exponential decay of u_0 from (2.5) yield

$$\int_{\mathbb{R}^n} |F_i(u+h,x) - F_i(u,x) - hF_i'(u,x)|\,dx \leq c(\|h\|^2 + \|h\|^{2p}), \qquad i = 1, 2,$$

for all $u, h \in H^1(\mathbb{R}^n)$ which shows that the functionals J_1, J_2 are Fréchet-differentiable.

Continuity of the Fréchet-derivative: Again we only need to consider J_1' and J_2'. By the mean value theorem we get for $i = 1, 2$

$$\begin{aligned}
|F_i'(s,x) - F_i'(t,x)| &\leq c\,\big||s+u_0(x)|^{p-1}(s+u_0(x)) - |t+u_0(x)|^{p-1}(t+u_0(x))\big| \\
&= c\,|s-t||\sigma+u_0(x)|^{p-1} \quad \text{ for } \sigma \text{ between } s,t \\
&\leq c|s-t|(|s|^{p-1} + |t|^{p-1} + |x|^{-2}).
\end{aligned} \tag{2.20}$$

Hence, if (u_j) converges to u in $H^1(\mathbb{R}^n)$ and if $\varphi \in H^1(\mathbb{R}^n)$ with $\|\varphi\| = 1$ then

$$\begin{aligned}
|J_i'[u_j](\varphi) - J_i'[u](\varphi)| &\leq c\int_{\mathbb{R}^n}\left(|u|^{p-1} + |u_j|^{p-1} + |x|^{-2}\right)|u_j - u||\varphi|\,dx \\
&\leq c(\|u\|_{L^{p+1}(\mathbb{R}^n)}^{p-1} + \|u_j\|_{L^{p+1}(\mathbb{R}^n)}^{p-1})\|u_j - u\|_{L^{p+1}(\mathbb{R}^n)}\|\varphi\|_{L^{p+1}(\mathbb{R}^n)} \\
&\quad + c\|u_j - u\|\|\varphi\| \\
&\leq c(\|u\|^{p-1} + \|u_j\|^{p-1} + 1)\|u_j - u\|
\end{aligned} \tag{2.21}$$

where we have used a triple Hölder-inequality, Hardy's inequality and Sobolev's embedding theorem. This shows $J_i'[u_j] \to J_i'[u]$ which finishes the proof. $\square$

Remark 7. *Note that in the case $n \geq 3, \alpha < \frac{n-6}{2}$ the integral $\int_{B_\rho} V(x)|x|^{-\frac{2}{p-1}} u\,dx$ is not well-defined for all $u \in H^1(\mathbb{R}^n)$ and all $p > \frac{n}{n-2}$. Indeed, if $V(x) = |x|^\alpha$ near the origin and $\alpha < \frac{n-6}{2}$ then we can find $p > \frac{n}{n-2}$ and $u \in H^1(\mathbb{R}^n)$ such that $\int_{B_\rho} |V(x)||x|^{-\frac{2}{p-1}}|u|\,dx = +\infty$, e.g. choose $u(x) = |x|^{\frac{2}{p-1}-n-\alpha} e^{-|x|^2} \in H^1(\mathbb{R}^n)$ for $p \in (\frac{n}{n-2}, \frac{2\alpha+n+6}{2\alpha+n+2})$ if $-\frac{n+2}{2} < \alpha < \frac{n-6}{2}$ and $p \in (\frac{n}{n-2}, \infty)$ in the case $\alpha \leq -\frac{n+2}{2}$.*

Proposition 8. *Let the assumptions of Theorem 2 hold. Then there exist values $\varepsilon, m, r_0 > 0$ such that for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$*

$$J[u] \geq m \quad \textit{for all } u \in H^1(\mathbb{R}^n) \textit{ with } \|u\| = r_0.$$

Proof. The choice of $\varepsilon, m, r_0 > 0$ stems from the estimate

$$J[u] \geq A(p)\|u\|^2 - B\|u\|^{p+1} - C(p)\|u\| \tag{2.22}$$

where $A(p) \to A > 0$ for some $A > 0$, $B > 0$ and $C(p) \to 0$ as $p \searrow \frac{n}{n-2}$. Let us first finish the proof assuming that (2.22) has already been shown.

Choice of ε, m, r_0: Let $r_0 := \min\{(\frac{A}{8B})^{\frac{1}{q-1}} : \frac{n}{n-2} \leq q \leq \frac{n+2}{n-2}\}$ and $m := \frac{A}{4} r_0^2$. We choose $\varepsilon > 0$ so small that for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ one has $A(p) \geq \frac{A}{2}$ and $C(p) \leq \frac{A}{8} r_0$. Then for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ and all $u \in H^1(\mathbb{R}^n)$ with $\|u\| = r_0$ we have

$$A(p)\|u\|^2 - B\|u\|^{p+1} - C(p)\|u\| \geq \frac{A}{2} r_0^2 - B r_0^{p+1} - C(p) r_0 \geq r_0^2 (\frac{A}{2} - \frac{A}{8} - \frac{A}{8}) = m$$

which gives the result.

It remains to prove (2.22). Let $A > 0$ be a constant such that $\|\cdot\|_V^2 \geq 2A\|\cdot\|^2$ on $H^1(\mathbb{R}^n)$ (see Remark 4). Using the estimates (2.15), (2.16) we get

$$|J_1[u]| + |J_2[u]| \leq c\,(\|u\|^{p+1} + c_{n,p}^{p-1}\|u\|^2 + c_{n,p}^p \|u\|).$$

From Proposition 5, (2.18) and the trace theorem we obtain

$$\begin{aligned} |J_3[u]| &\leq \int_{B_\rho} |V(x) u_0 u|\,dx + \int_{\mathbb{R}^n \setminus B_\rho} |(V(x)-1) u_0 u|\,dx + \int_{\partial B_\rho} |\partial_\nu^+ u_0 - \partial_\nu^- u_0||\gamma(u)|\,d\sigma \\ &\leq c\, c_{n,p}(D(p)+1)\|u\|, \end{aligned}$$

where the value $D(p)$ is defined in (2.17). This results in the estimate

$$\begin{aligned} J[u] &\geq \frac{1}{2}\|u\|_V^2 - |J_1[u]| - |J_2[u]| - |J_3[u]| \\ &\geq \underbrace{(A - c\, c_{n,p}^{p-1})}_{=:A(p)} \|u\|^2 - c\|u\|^{p+1} - \underbrace{c\big(c_{n,p}^p + c_{n,p}(D(p)+1)\big)}_{=:C(p)} \|u\|. \end{aligned}$$

Clearly, $A(p) \to A$ as $p \searrow \frac{n}{n-2}$. Furthermore, $C(p) \to 0$ as $p \searrow \frac{n}{n-2}$. Indeed, if $\alpha > \frac{n-6}{2}$ then (2.17) shows that $D(p)$ is uniformly bounded in p for $p > \frac{n}{n-2}$. If $\alpha = \frac{n-6}{2}$ then $D(p) \to \infty$ but still $c_{n,p}D(p) \to 0$ as $p \searrow \frac{n}{n-2}$. This finally proves (2.22). $\square$

Now we look for a critical point within $\{u \in H^1(\mathbb{R}^n) : \|u\| < r_0\}$. We recall Ekeland's variational principle, cf. Struwe [26], Theorem 5.1.

Ekeland's variational principle. *Let M be a complete metric space with metric d, and let $J : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, bounded from below, and $\not\equiv \infty$. Then, for any $\eta, \delta > 0$, and $u \in M$ with*

$$J[u] \le \inf_M J + \eta$$

there is an element $w \in M$ strictly minimizing the functional

$$J_w[z] \equiv J[z] + \frac{\eta}{\delta} d(w, z).$$

Moreover, we have $J[w] \le J[u]$ and $d(w, u) \le \delta$.

Proposition 9. *Let the assumptions of Theorem 2 hold and let $\varepsilon, m, r_0 > 0$ be the values from Proposition 8. Then for all $p \in (\frac{n}{n-2}, \frac{n}{n-2} + \varepsilon)$ the functional J has a nontrivial critical point $\tilde{u} \in H^1(\mathbb{R}^n)$ with $\|\tilde{u}\| \le r_0$.*

Proof. *Step 1:* Let us find a weakly convergent Palais-Smale sequence. Consider the minimization problem

$$\inf_M J \quad \text{where } M = \{u \in H^1(\mathbb{R}^n) : \|u\| \le r_0\}.$$

Choose a positive sequence $\eta_j \to 0$ as $j \to \infty$ and let $\tilde{u}_j \in M$ be such that

$$J[\tilde{u}_j] \le \inf_M J + \eta_j^2.$$

Using Ekeland's variational principle with $\eta = \eta_j^2$ and $\delta = \eta_j$ we find $u_j \in M$ such that

$$J[u_j] \le J[z] + \eta_j \|z - u_j\| \quad \text{for all } z \in M.$$

Then (u_j) is also a minimizing sequence for $J|_M$ and since $0 \in M$ and $J[0] = 0 < m$ we see that $\|u_j\| < r_0$ for large j. Hence, almost all u_j are interior points of M. Applying the estimate

$$\begin{aligned} J[z] &= J[u_j] + J'[u_j](z - u_j) + o(\|z - u_j\|) \\ &\le J[z] + J'[u_j](z - u_j) + \eta_j \|z - u_j\| + o(\|z - u_j\|) \quad \text{as } z \to u_j, z \in M \end{aligned}$$

to $z = u_j + tv$ with $\|v\| = 1$ we find for $t \to 0$

$$\|J'[u_j]\| = \sup_{\|v\|=1} |J'[u_j](v)| \le \eta_j \to 0 \quad \text{as } j \to \infty,$$

i.e. (u_j) is a minimizing Palais-Smale sequence of $J|_M$. Moreover, since (u_j) is bounded in $H^1(\mathbb{R}^n)$ by r_0 we may assume (up to selecting subsequences) that $u_j \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^n)$ and $u_j \to \tilde{u}$ almost everywhere in $\mathbb{R}^n$.

Step 2: Let us show that the weak limit $\tilde u$ is a critical point of J. So let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a fixed test function, $K := \operatorname{supp}(\varphi)$. Because of $u_j \to \tilde u$ in $L^{p+1}(K)$ by compact embedding we may use Lemma A.1 in [31] to find a function $w_\varphi \in L^{p+1}(K)$ and a subsequence (possibly depending on φ) again denoted by (u_j) such that $|\tilde u|, |u_j| \le w_\varphi$. Recalling (2.20) we get

$$|J_i'[u_j]\varphi - J_i'[\tilde u]\varphi| \le c \int_K \big(w_\varphi^{p-1} + \frac{1}{|x|^2}\big)|u_j - \tilde u||\varphi|\,dx \quad \text{ for } i = 1,2 \text{ and } j \in \mathbb{N}.$$

The integrand is pointwise almost everywhere bounded by $2w_\varphi^p|\varphi| + \frac{2}{|x|^2}w_\varphi|\varphi|$. Since $w_\varphi \in L^{p+1}(K), \varphi \in L^\infty(K)$ and $|x|^{-2} \in L^{\frac{p+1}{p}}(K)$ the dominated convergence theorem applies and yields

$$J_i'[u_j](\varphi) \to J_i'[\tilde u](\varphi) \quad \text{ for } i = 1,2 \text{ as } j \to \infty.$$

Weak convergence implies $\langle u_j, \varphi\rangle_V \to \langle \tilde u, \varphi\rangle_V$. Furthermore $J_3'[u_j](\varphi) = J_3'[\tilde u](\varphi) = J_3[\varphi]$ by linearity. In total we see that $J'[\tilde u](\varphi) = \lim_{j\to\infty} J'[u_j](\varphi) = 0$ for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ which proves the result. □

2.4. **The distributional solution property.** In Proposition 9 we have proved that under the assumptions of Theorem 2 a critical point $\tilde u \in H^1(\mathbb{R}^n)$ of J exists provided $\varepsilon > 0$ is sufficiently small. Due to the properties of u_0 (cf. Proposition 5) we find that $U = u_0 + \tilde u$ lies in $H^1(\mathbb{R}^n \setminus B_\delta)$ for every $\delta > 0$ and $U \in L^q_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{n(p-1)}{2})$. From part (iii) of Theorem 2 which is proved in the next section we get $U \in L^q(\mathbb{R}^n)$ for all $q \in [1, \frac{n(p-1)}{2})$. Since the Euler-equation (2.8) for $\tilde u$ and equation (2.7) hold for all $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we obtain that for every $\delta > 0$ the function $U = u_0 + \tilde u$ is a weak solution of (1.1) on $\mathbb{R}^n \setminus B_\delta$.

In order to complete the proof of Theorem 2,(i),(ii) it therefore remains to show that U is an unbounded distributional solution of (1.1).

Lemma 10. *Let the assumptions of Theorem 2 hold and let $\tilde u \in H^1(\mathbb{R}^n)$ be a critical point of J according to Proposition 9. Then the function $U := u_0 + \tilde u$ is a distributional solution of* (1.1) *with* $\operatorname{ess\,sup}_{B_\delta} U = +\infty$ *for all $\delta > 0$.*

Proof. According to the definition of u_0 for all $\delta > 0$:

$$\int_{B_\delta} |u_0(x)|\,dx = O(\delta^{-\frac{2}{p-1}+n}), \qquad \int_{B_\delta} |u_0(x)|^p\,dx = O(\delta^{-\frac{2p}{p-1}+n}),$$
$$\oint_{\partial B_\delta} |u_0(x)|\,dx = O(\delta^{-\frac{2}{p-1}+n-1}), \qquad \oint_{\partial B_\delta} |\partial_\nu^\pm u_0(x)|\,dx = O(\delta^{-\frac{p+1}{p-1}+n-1}).$$

All integrals converge to 0 as $\delta \to 0$ since $p > \frac{n}{n-2} > \frac{n+1}{n-1} > \frac{n+2}{n}$. Hence, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ we find from Proposition 5,(i)

$$\begin{aligned}\int_{B_\rho} u_0(-\Delta\varphi)\,dx &= \lim_{\delta\to 0}\int_{B_\rho\setminus B_\delta} u_0(-\Delta\varphi)\,dx \\ &= \lim_{\delta\to 0}\int_{B_\rho\setminus B_\delta} (-\Delta u_0)\varphi\,dx - \oint_{\partial B_\rho} (u_0\partial_\nu^+\varphi - \varphi\partial_\nu^+ u_0)\,d\sigma\end{aligned}$$

$$= \int_{B_\rho} u_0^p \varphi \, dx - \oint_{\partial B_\rho} (u_0 \partial_\nu^+ \varphi - \varphi \partial_\nu^+ u_0) \, d\sigma \tag{2.23}$$

and since φ has compact support Proposition 5,(ii) implies

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\rho} u_0(-\Delta\varphi) \, dx &= \int_{\mathbb{R}^n \setminus B_\rho} (-\Delta u_0)\varphi \, dx + \oint_{\partial B_\rho} (u_0 \partial_\nu^- \varphi - \varphi \partial_\nu^- u_0) \, d\sigma \\ &= \int_{\mathbb{R}^n \setminus B_\rho} (u_0^p - u_0)\varphi \, dx + \oint_{\partial B_\rho} (u_0 \partial_\nu^- \varphi - \varphi \partial_\nu^- u_0) \, d\sigma. \end{aligned} \tag{2.24}$$

Since φ is smooth we have $\partial_\nu^- \varphi = \partial_\nu^+ \varphi$ on ∂B_ρ. Using (H1) we find $V u_0 \in L^1_{loc}(\mathbb{R}^n)$ by direct calculation. Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} u_0(-\Delta\varphi + V(x)\varphi) \, dx &= \int_{\mathbb{R}^n} u_0^p \varphi \, dx + \int_{B_\rho} V(x) u_0 \varphi \, dx \\ &\quad + \int_{\mathbb{R}^n \setminus B_\rho} (V(x) - 1) u_0 \varphi \, d\sigma + \oint_{\partial B_\rho} (\partial_\nu^+ u_0 - \partial_\nu^- u_0)\varphi \, d\sigma. \end{aligned} \tag{2.25}$$

On the other hand $\tilde{u}$ is a critical point of J and thus satisfies the Euler-equation (2.8) for all $\varphi \in H^1(\mathbb{R}^n)$. Moreover, $V\tilde{u} \in L^1_{loc}(\mathbb{R}^n)$ and hence, $VU = Vu_0 + V\tilde{u} \in L^1_{loc}(\mathbb{R}^n)$. Adding up (2.8) and (2.25) gives

$$\int_{\mathbb{R}^n} U(-\Delta\varphi + V(x)\varphi) \, dx = \int_{\mathbb{R}^n} \Gamma(x)|U|^{p-1} U \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n).$$

Hence, U is a distributional solution of (1.1).

Now assume $U \leq C_\delta < \infty$ almost everywhere on B_δ for some $\delta > 0$. Choosing $\delta' \in (0, \delta)$ such that $u_0(x) \geq 2C_\delta$ on $B_{\delta'}$ (see Proposition 5,(iv)) we get $\tilde{u} = U - u_0 \leq -\frac{u_0}{2} < 0$ almost everywhere on $B_{\delta'}$ and thus

$$\|\tilde{u}\|_{L^{\frac{2n}{n-2}}(B_{\delta'})} \geq \frac{1}{2} \|u_0\|_{L^{\frac{2n}{n-2}}(B_{\delta'})} = +\infty$$

which contradicts $\tilde{u} \in H^1(\mathbb{R}^n)$. Hence, $\operatorname{ess\,sup}_{B_\delta} U = +\infty$. □

Remark 11. *Clearly,* $u_0 \notin H^1(B_1)$ *so that* $U := u_0 + \tilde{u} \notin H^1(\mathbb{R}^n)$.

2.5. **Exponential decay.** Let us prove part (iii) of Theorem 2. For the reader's convenience we only present the main idea of the proof, details are given in Appendix B.

Lemma 12. *Let the assumptions of Theorem 2 hold and let* $\tilde{u} \in H^1(\mathbb{R}^n)$ *be a critical point of* J *according to Proposition 9, let* $U := u_0 + \tilde{u}$. *Then for all* $0 < \mu < \sqrt{\Sigma}$ *there is* $C_\mu > 0$ *such that* $|U(x)| \leq C_\mu e^{-\mu|x|}$ *for all* $x \in \mathbb{R}^n$ *with* $|x| \geq 1$.

Proof. Applying Proposition 20 to $u = U$, $\Omega = \mathbb{R}^n \setminus B_2, q = p$ and $W := V - \Gamma|U|^{p-1} 1_{\mathbb{R}^n \setminus B_2}$ we deduce that U can be assumed to be continuous and that we have $U(x) \to 0$ as $|x| \to \infty$. Note that $W \in L^\infty(\mathbb{R}^n \setminus B_1) + L^{\frac{2n}{(n-2)(p-1)}}(\mathbb{R}^n \setminus B_1) \subset K_n(\mathbb{R}^n \setminus B_2)$ due to $\frac{2n}{(n-2)(p-1)} > \frac{n}{2}$ and (1.8). From $U(x) \to 0$ as $|x| \to \infty$ and [20], Theorem 8.3.1 we obtain

$$\sigma_{ess}(-\Delta + W) = \sigma_{ess}(-\Delta + V) \subset [\Sigma, \infty).$$

Then Proposition 21 applied to $\Omega = \mathbb{R}^n \setminus B_2$, $s = \frac{2n}{(n-2)(p-1)}$, $q = 2$ gives $|U(x)| \leq C'_\mu e^{-\mu|x|}$ for all $x \in \mathbb{R}^n$ with $|x| \geq 3$. Since $U \in H^1(\mathbb{R}^n \setminus B_\delta)$ satisfies a subcritical elliptic PDE in $\mathbb{R}^n \setminus B_\delta$ for all $\delta > 0$ the result follows from the DeGiorgi-Nash-Moser local boundedness principle. □

2.6. **Positivity in the case $\Gamma \geq 0$.** In this section we prove part (iv) of Theorem 2, so let us assume $\Gamma \geq 0$. As pointed out before (see (2.10) and the following remarks) the results of the previous sections 2.3, 2.4, 2.5 also apply to $\hat{J}$, in particular we find a critical point $\hat{u}$ of $\hat{J}$. By Lemma 10 the function $\hat{U} = u_0 + \hat{u}$ satisfies $\operatorname{ess\,sup}_{B_\delta} \hat{U} = +\infty$ for all $\delta > 0$ and is a distributional solution of

$$\int_{\mathbb{R}^n} \hat{U}(-\Delta\varphi + V(x)\varphi)\,dx = \int_{\mathbb{R}^n} \Gamma(x)\hat{U}_+^p \varphi\,dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n).$$

It remains to show that $\hat{U}$ must be positive.

To this end let $\psi \in C_0^\infty(\mathbb{R}^n), \psi \geq 0$ be arbitrary, set $K := \operatorname{supp}(\psi)$. Let then $w \in H^1(\mathbb{R}^n)$ be the unique weak solution of $-\Delta w + V(x)w = \psi$ obtained by minimizing the functional $L[z] := \int_{\mathbb{R}^n} |\nabla z|^2 + V(x)z^2 - 2\psi z\,dx$ over $H^1(\mathbb{R}^n)$.

Since $\psi \geq 0$ one sees that $w \geq 0$ (if w is a minimizer then also $|w|$ is a minimizer and L has a unique minimizer). Then $-\Delta w = f$ in the weak sense where $f = \psi - Vw$ and $V \in L^q_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{2n}{(6-n)_+})$. From (H1) and $w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ we infer $f \in L^q_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{n}{2})$ so that Caldéron-Zygmund estimates (cf. Chapter 9 in [6]) imply $w \in W^{2,q}_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{n}{2})$. Sobolev's imbedding theorem then implies $f \in L^q_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{2n}{(6-n)_+})$ and thus $w \in W^{2,q}_{loc}(\mathbb{R}^n)$ for all $q \in [1, \frac{2n}{(6-n)_+})$ again by Caldéron-Zygmund estimates. In particular, up to a set of measure zero w is locally uniformly continuous and satisfies $-\Delta w + Vw = \psi$ pointwise in $\mathbb{R}^n$.

Since $p > \frac{n}{n-2}$ we can find $s \in (\frac{n(p-1)}{n(p-1)-2}, \frac{2n}{(6-n)_+})$. Recall from Section 2.4 that this choice of s implies $\hat{U} \in L^{\frac{s}{s-1}}(K)$. Let (φ_k) be a sequence of positive $C_0^\infty(\mathbb{R}^n)$-functions such that $\varphi_k \to w$ uniformly on K and in $W^{2,s}(K)$. Then $\hat{U}V \in L^1(K)$ and

$$\begin{aligned}
\int_{\mathbb{R}^n} \hat{U}(x)\psi(x)\,dx &= \int_K \hat{U}(x)\big(-\Delta w + V(x)w\big)\,dx \\
&= \lim_{k\to\infty} \int_K \hat{U}(x)\big(-\Delta\varphi_k + V(x)\varphi_k\big)\,dx \\
&= \lim_{k\to\infty} \int_K \Gamma(x)\hat{U}(x)_+^p \varphi_k(x)\,dx \\
&= \int_K \Gamma(x)\hat{U}(x)_+^p w(x)\,dx \geq 0.
\end{aligned}$$

Since $\psi \in C_0^\infty(\mathbb{R}^n), \psi \geq 0$ is arbitrary we obtain $\hat{U} \geq 0$ almost everywhere. □

3. Proof of Theorem 3

Under the assumptions of Theorem 3 we now prove regularity properties of distributional solutions of (1.2) in the case $1 < p < \frac{n}{(n-2)_+}$. For $\omega > 0$ we rewrite (1.2) in the following way

$$-\Delta u + \omega u = g_\omega \quad \text{where } g_\omega(x) := g(x, u(x)) + (\omega - V(x))u(x). \tag{3.1}$$

We will show that (3.1) can be written in form of an integral equation using the Green function G_ω of $-\Delta + \omega$. Therefore we are lead to study the operator T_ω given by

$$T_\omega(f) := \int_{\mathbb{R}^n} G_\omega(x-y) f(y)\, dy.$$

It is well-known (cf. [7], [24]) that

$$G_\omega(x) = \omega^{\frac{n-2}{2}} G_1(\sqrt{\omega}x) = (2\pi)^{-\frac{n}{2}} |\omega^{-1/2} x|^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(\sqrt{\omega}|x|). \tag{3.2}$$

The following expansions can be found in [7] for multiindices α with $|\alpha| \geq 1$:

$$G_\omega(x) = \begin{cases} O(1), & n = 1 \\ O(\log \frac{1}{|x|}), & n = 2 \\ O(|x|^{2-n}), & n \geq 3 \end{cases} \quad \text{as } |x| \to 0, \quad G_\omega(x) = O(e^{-\sqrt{\omega}|x|}) \text{ as } |x| \to \infty. \tag{3.3}$$

$$D^\alpha G_\omega(x) = O(|x|^{2-n-|\alpha|}) \quad \text{as } |x| \to 0, \quad D^\alpha G_\omega(x) = O(e^{-\sqrt{\omega}|x|}) \text{ as } |x| \to \infty. \tag{3.4}$$

The proof of Theorem 3 is given in three steps: In Proposition 13 and Proposition 14 we study the mapping properties of T_ω for fixed $\omega > 0$ in order to prove in Proposition 16 the representation formula $u = T_\omega(g_\omega)$ for every distributional solution u of (1.2) with $u \in L^p(\mathbb{R}^n; \omega_0)$ and $\omega_0 < \omega$. Finally we obtain the regularity result of Theorem 3 by a combination of the mapping properties of T_ω with the continuity/decay results of Proposition 20 and Proposition 21.

Proposition 13. *Let $\omega > 0$, $k \in \{0, 1, 2\}$ and $q, r \in [1, \infty]$. Then*

$$T_\omega : L^q(\mathbb{R}^n) \to W^{k,r}(\mathbb{R}^n)$$

provided $s := (1 + \frac{1}{r} - \frac{1}{q})^{-1}$ satisfies one of the following conditions:

(i) *Case $k = 0$: $s \in [1, \frac{n}{(n-2)_+})$ or $n = 1, s = \infty$ or $n \geq 3, q \in (1, \frac{n}{2}), s = \frac{n}{n-2}$.*
(ii) *Case $k = 1$: $s \in [1, \frac{n}{(n-1)_+})$ or $n = 1, s = \infty$ or $n \geq 2, q \in (1, n), s = \frac{n}{n-1}$.*
(iii) *Case $k = 2$: $q = r \in (1, \infty)$.*

In each case there exists a constant $c = c(k, q, r, n) > 0$ such that

$$\|T_\omega f\|_{W^{k,r}(\mathbb{R}^n)} \leq c\|f\|_{L^q(\mathbb{R}^n)} \quad \text{for all } f \in L^q(\mathbb{R}^n).$$

Furthermore, in the cases $k = 1$ or $k = 2$ we have for all $|\alpha| = 1$

$$D^\alpha(T_\omega f)(x) = \int_{\mathbb{R}^n} (D^\alpha G_\omega)(x-y) f(y)\, dy.$$

Proof. The proof of (iii) can be found in [24], Theorem 3, Chapter V. Let us prove (i), i.e, $k=0$. Young's inequality gives

$$\|T_\omega f\|_{L^r(\mathbb{R}^n)} = \|G_\omega * f\|_{L^r(\mathbb{R}^n)} \leq \|G_\omega\|_{L^s(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)}$$

provided $q,r,s \in [1,\infty]$ satisfy $1+\frac{1}{r} = \frac{1}{s}+\frac{1}{q}$. In the cases $n=1, n\geq 2$ the asymptotic formulas (3.3), (3.4) show that $G_\omega \in L^s(\mathbb{R}^n)$ for all $s\in[1,\infty], [1,\frac{n}{n-2})$ respectively and the first two subcases are proved. The case $n\geq 3, q\in(1,\frac{n}{2}), s=\frac{n}{n-2}$ follows from (iii) and Sobolev's imbedding theorem $W^{2,q}(\mathbb{R}^n)\to L^{\frac{nq}{n-2q}}(\mathbb{R}^n)$.

Next we prove (ii). By (3.3), (3.4) we have $|\nabla G_\omega(z)| \sim |z|^{1-n}$ as $z\to 0$ and $|\nabla G_\omega(z)| \sim e^{-\sqrt{\omega}|z|}$ as $|z|\to\infty$. Hence $|\nabla G_\omega| \in L^s(\mathbb{R}^n)$ for $s\in[1,\infty], [1,\frac{n}{n-1})$ in the cases $n=1, n\geq 2$ respectively. In these cases the dominated convergence theorem and Young's inequality apply and yield $\nabla(T_\omega f) = \nabla G_\omega * f$ as well as

$$\||\nabla(T_\omega f)|\|_{L^r(\mathbb{R}^n)} \leq \||\nabla G_\omega|\|_{L^s(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)}.$$

The case $n\geq 2, q\in(1,n), s=\frac{n}{n-1}$ again follows from the case $k=2$ and Sobolev's imbedding theorem $W^{2,q}(\mathbb{R}^n)\to W^{1,\frac{nq}{n-q}}(\mathbb{R}^n)$. □

We will also need the following local version of Proposition 13 where we use weighted Lebesgue spaces

$$L^q(\mathbb{R}^n;\omega) := \big\{u\in L^q_{loc}(\mathbb{R}^n) \ :\ \int_{\mathbb{R}^n} |u(x)|^q e^{-\sqrt{\omega}|x|}\,dx < \infty\big\}. \tag{3.5}$$

for $1\leq q<\infty$ and $\omega>0$. We set $\|u\|_{L^q(\mathbb{R}^n;\omega)} := \big(\int_{\mathbb{R}^n} |u(x)|^q e^{-\sqrt{\omega}|x|}\,dx\big)^{1/q}$.

Proposition 14. *Let $\omega>0$, $k\in\{0,1\}$ and $q,r\in[1,\infty]$. Then*

$$T_\omega : L^q_{loc}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n;\omega) \to W^{k,r}_{loc}(\mathbb{R}^n)$$

provided $s:=(1+\frac{1}{r}-\frac{1}{q})^{-1}$ satisfies one of the following conditions:

(i) *Case $k=0$: $s\in[1,\frac{n}{(n-2)_+})$ or $n=1, s=\infty$.*
(ii) *Case $k=1$: $s\in[1,\frac{n}{(n-1)_+})$ or $n=1, s=\infty$.*

In each case for all compact sets K_1, K_2 such that $K_1 \subset\subset K_2$ there exists a constant $c=c(k,q,r,n,K_1,K_2)>0$ such that

$$\|T_\omega f\|_{W^{k,r}(K_1)} \leq c\big(\|f\|_{L^q(K_2)} + \|f\|_{L^1(\mathbb{R}^n;\omega)}\big) \quad \textit{for all } f\in L^q_{loc}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n;\omega)$$

First order derivatives of $T_\omega f$ can be taken under the integral as in Proposition 13.

Proof. Consider first the case $k=0$. For given compact sets K_1, K_2 with $K_1\subset\subset K_2$ let B be an open ball centered at 0 such that $K_1+\overline{B}\subset K_2$ where $K_1+\overline{B}$ denotes the Minkowski sum of K_1 and $\overline{B}$. Then there exists $C_B>0$ such that $|G_\omega(z)|\leq C_B e^{-\sqrt{\omega}|z|}$ for all $z\in\mathbb{R}^n\setminus B$, cf. (3.3). If q,r are as in the theorem with $r<\infty$ then Proposition 13,(i) shows

$$\|T_\omega f\|^r_{L^r(K_1)} \leq \int_{K_1}\left(\int_{x+B} G_\omega(x-y)|f(y)|\,dy + \int_{x+\mathbb{R}^n\setminus B} G_\omega(x-y)|f(y)|\,dy\right)^r dx$$

$$\begin{aligned}
&\le c\left(\|T_\omega(|f\,1_{K_2}|)\|^r_{L^r(\mathbb{R}^n)} + C_B^r \int_{K_1} \big(\int_{\mathbb{R}^n} e^{\sqrt{\omega}(|x|-|y|)}|f(y)|\,dy\big)^r\,dx\right)\\
&\le c\left(\|f\,1_{K_2}\|^r_{L^q(\mathbb{R}^n)} + \big(\int_{\mathbb{R}^n} e^{-\sqrt{\omega}|y|}|f(y)|\,dy\big)^r\right)\\
&= c(\|f\|^r_{L^q(K_2)} + \|f\|^r_{L^1(\mathbb{R}^n;\omega)}).
\end{aligned}$$

In the case $r=\infty$ we obtain with the same notations as above

$$\begin{aligned}
\|T_\omega f\|_{L^\infty(K_1)} &\le c\left(\|T_\omega(|f\,1_{K_2}|)\|_{L^\infty(\mathbb{R}^n)} + \|\int_{\mathbb{R}^n} e^{\sqrt{\omega}(|\cdot|-|y|)}|f(y)|\,dy\|_{L^\infty(K_1)}\right)\\
&\le c\left(\|f\|_{L^q(K_2)} + \|f\|_{L^1(\mathbb{R}^n;\omega)}\right).
\end{aligned}$$

This finishes the proof of (i). The case $k=1$ is treated similarly using the mapping property (ii) in Proposition 13 instead of (i). □

Next we prove the representation formula $u = T_\omega(g_\omega)$ for distributional solutions u of (1.2) which satisfy $u \in L^p(\mathbb{R}^n;\omega_0)$ for some $\omega_0 < \omega$. To this end we first show that the corresponding linear problem has at most one solution in $L^1(\mathbb{R}^n;\omega)$.

Proposition 15. *Let $v \in L^1(\mathbb{R}^n;\omega)$ be a distributional solution of $-\Delta v + \omega v = 0$. Then $v=0$.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be arbitrary and for $R>0$ set $\varphi_R := \chi_R T_\omega(\psi) \in C_0^\infty(\mathbb{R}^n)$ where $\chi_R(x) = \chi(R^{-1}x)$ for a fixed function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(0)=1$. Since $v \in L^1(\mathbb{R}^n;\omega)$ we have $|T_\omega(\psi)||v| + |\nabla T_\omega(\psi)||v| \in L^1(\mathbb{R}^n)$. Hence the dominated convergence theorem gives

$$\begin{aligned}
0 &= \lim_{R\to\infty} \int_{\mathbb{R}^n} v(-\Delta\varphi_R + \omega\varphi_R)\,dx\\
&= \lim_{R\to\infty} \Big[\int_{\mathbb{R}^n} \chi_R v\psi\,dx + \int_{\mathbb{R}^n} \big(-\Delta\chi_R T_\omega(\psi) - 2\nabla\chi_R\nabla T_\omega(\psi)\big)v\,dx\Big]\\
&= \int_{\mathbb{R}^n} v\psi\,dx.
\end{aligned}$$

Since $\psi \in C_0^\infty(\mathbb{R}^n)$ was arbitrary we get $v=0$. □

Proposition 16. *Let $1 \le p < \infty$ and let $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $|g(x,s)| \le C(1+|s|^p)$ for all $s \in \mathbb{R}$ and almost all $x \in \mathbb{R}^n$. Let $u \in L^p(\mathbb{R}^n;\omega_0)$ for some $\omega_0 > 0$ be a distributional solution of (1.2). Then for all $\omega > \omega_0$ we have $u = T_\omega(g_\omega)$ almost everywhere on $\mathbb{R}^n$ with g_ω given by (3.1).*

Proof. By assumption the function $u \in L^p(\mathbb{R}^n;\omega_0) \subset L^1(\mathbb{R}^n;\omega)$ satisfies

$$\int_{\mathbb{R}^n} u(-\Delta\varphi + \omega\varphi)\,dx = \int_{\mathbb{R}^n} g_\omega\varphi\,dx \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

On the other hand let us show that $T_\omega(g_\omega) \in L^1(\mathbb{R}^n;\omega)$ satisfies the same integral relation. Indeed, we have $g_\omega = g(\cdot,u) + (\omega - V)u \in L^1(\mathbb{R}^n;\omega_0)$ so that (3.3) implies

$$\int_{\mathbb{R}^n} |T_\omega(g_\omega)|e^{-\sqrt{\omega}|x|}\,dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_\omega(x-y)|g_\omega(y)|e^{-\sqrt{\omega}|x|}\,dx\,dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} |g_\omega(y)| e^{-\sqrt{\omega_0}|y|} \int_{\mathbb{R}^n} e^{\sqrt{\omega_0}|y|} G_\omega(x-y) e^{-\sqrt{\omega}|x|}\,dx\,dy \\
&\leq \int_{\mathbb{R}^n} |g_\omega(y)| e^{-\sqrt{\omega_0}|y|} \Big[c \int_{\{|x-y|\geq 1\}} e^{\sqrt{\omega_0}|y|} e^{-\sqrt{\omega}|x-y|} e^{-\sqrt{\omega}|x|}\,dx \\
&\qquad + \int_{\{|x-y|\leq 1\}} e^{\sqrt{\omega_0}|y|} G_\omega(x-y) e^{-\sqrt{\omega}|x|}\,dx \Big]\,dy \\
&\leq \int_{\mathbb{R}^n} |g_\omega(y)| e^{-\sqrt{\omega_0}|y|} \Big[c \int_{\{|x-y|\geq 1\}} e^{\sqrt{\omega_0}|y|} e^{-\sqrt{\omega_0}|x-y|} e^{-\sqrt{\omega}|x|}\,dx \\
&\qquad + \int_{\{|x-y|\leq 1\}} e^{\sqrt{\omega_0}|y|} G_\omega(x-y) e^{-\sqrt{\omega}(|y|-1)}\,dx \Big]\,dy \\
&\leq c \int_{\mathbb{R}^n} |g_\omega(y)| e^{-\sqrt{\omega_0}|y|} \Big[\int_{\{|x-y|\geq 1\}} e^{(\sqrt{\omega_0}-\sqrt{\omega})|x|}\,dx + \int_{\{|z|\leq 1\}} G_\omega(z)\,dz \Big]\,dy \\
&\leq c \int_{\mathbb{R}^n} |g_\omega(y)| e^{-\sqrt{\omega_0}|y|}\,dy < \infty,
\end{aligned}
$$

where we have used that G_ω is a locally integrable function. Furthermore, Fubini's theorem yields for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$
\begin{aligned}
\int_{\mathbb{R}^n} T_\omega(g_\omega)(-\Delta\varphi + \omega\varphi)\,dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G_\omega(x-y) g_\omega(y)\,dy \right) (-\Delta\varphi(x) + \omega\varphi(x))\,dx \\
&= \int_{\mathbb{R}^n} g_\omega(y) \left(\int_{\mathbb{R}^n} G_\omega(x-y)(-\Delta\varphi(x) + \omega\varphi(x))\,dx \right) dy \\
&= \int_{\mathbb{R}^n} g_\omega(y) \left(\int_{\mathbb{R}^n} G_\omega(y-x)(-\Delta\varphi(x) + \omega\varphi(x))\,dx \right) dy \\
&= \int_{\mathbb{R}^n} g_\omega(y)\varphi(y)\,dy. \qquad (3.6)
\end{aligned}
$$

Applying Proposition 15 to $v = u - T_\omega(g_\omega)$ we conclude $u = T_\omega(g_\omega)$. □

3.1. **Proof of Theorem 3,(2).** Let g satisfy (1.4) and let $u \in L^p(\mathbb{R}^n)$ be a distributional solution of (1.2). Then (1.4) and the assumption $1 < p < \frac{n}{n-2}$ implies that

$$
W(x) := V(x) - \frac{g(x,u(x))}{u(x)} 1_{\{u(x)\neq 0\}}
$$

lies in the Kato class K_n – see (1.8) – and thus Proposition 20 implies $u \in L^\infty(\mathbb{R}^n)$ and $u(x) \to 0$ as $|x| \to \infty$. Hence, $u \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and thus $g_\omega \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ where g_ω is defined in (3.1). From Proposition 16 we get $u = T_\omega(g_\omega)$. From Proposition 13 with $(k,q,r) = (1,q,q), q \in [p,\infty]$ and $(k,q,r) = (2,q',q'), q' \in [p,\infty)$ we get $u \in W^{1,q}(\mathbb{R}^n) \cap W^{2,q'}(\mathbb{R}^n)$ for all $q \in [p,\infty], q' \in [p,\infty)$.

Now, in addition let us assume (H2) and (1.5). Then [20], Theorem 8.3.1 implies

$$\sigma_{ess}(-\Delta + W(x)) = \sigma_{ess}(-\Delta + V(x)) \subset [\Sigma, \infty)$$

Hence, Proposition 21 applies to u and $\Omega = \mathbb{R}^n$ and it follows $|u(x)| \le C_\mu e^{-\mu|x|}$ for almost all $x \in \mathbb{R}^n$. In particular $u \in L^1(\mathbb{R}^n)$ so that $u \in W^{1,q}(\mathbb{R}^n) \cap W^{2,q'}(\mathbb{R}^n)$ for all $q \in [1,\infty], q' \in (1,\infty)$ by Proposition 13.

Finally, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we get from $u \in W^{2,1}_{loc}(\mathbb{R}^n)$ and the definition of a weak derivative

$$\int_{\mathbb{R}^n} (-\Delta u + Vu)\varphi \, dx = \int_{\mathbb{R}^n} u(-\Delta\varphi + V\varphi) \, dx = \int_{\mathbb{R}^n} g(x,u)\varphi \, dx,$$

hence $-\Delta u + Vu = g(x,u)$ almost everywhere which proves that u is a strong solution of (1.2).

□

3.2. **Proof of Theorem 3,(1).** Our aim is to show that u satisfies the assumptions of Proposition 16 so that we may infer the local regularity properties of u from the representation formula $u = T_\omega(g_\omega)$ and the mapping properties of T_ω. For our approach we first need to check that functions in $W_0^{2,\infty}(\mathbb{R}^n)$ with compact support are admissible test functions for (1.2).

Proposition 17. *Let the assumptions of Theorem 3,(1) hold. Then*

$$\int_{\mathbb{R}^n} u(-\Delta\varphi + V(x)\varphi) \, dx = \int_{\mathbb{R}^n} g(x,u)\varphi \, dx$$

for all $\varphi \in W_0^{2,\infty}(\mathbb{R}^n)$ *such that* $\operatorname{supp}(\varphi)$ *is compact.*

Proof. Let $\varphi \in W_0^{2,\infty}(\mathbb{R}^n)$ with compact support. By mollification we obtain a sequence $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ and a compact set K such that $\operatorname{supp}(\varphi), \operatorname{supp}(\varphi_k) \subset K$, $\Delta\varphi_k \to \Delta\varphi$ pointwise almost everywhere in K, $|\Delta\varphi_k| \le \|\Delta\varphi\|_\infty$ and $\varphi_k \to \varphi$ uniformly. The dominated convergence theorem gives

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x,u)\varphi \, dx &= \int_K g(x,u)\varphi \, dx \\
&= \lim_{k\to\infty} \int_K g(x,u)\varphi_k \, dx \\
&= \lim_{k\to\infty} \int_K u(-\Delta\varphi_k + V(x)\varphi_k) \, dx \\
&= \int_K u(-\Delta\varphi + V(x)\varphi) \, dx \\
&= \int_{\mathbb{R}^n} u(-\Delta\varphi + V(x)\varphi) \, dx.
\end{aligned}$$

□

In the following Proposition we verify the assumptions of Proposition 16 in order to deduce $u = T_\omega(g_\omega)$.

Proposition 18. *Let the assumptions of Theorem 3,(1) hold. Then $u \in L^p(\mathbb{R}^n;\omega_0)$ for all $\omega_0 > 0$.*

Proof. Let $J_{\frac{n-2}{2}}$ denote the Bessel function of the first kind of order $\frac{n-2}{2}$, let $v(r) := J_{\frac{n-2}{2}}(r)r^{\frac{2-n}{2}}$. Then v lies in $C^\infty(\mathbb{R})$ and is a classical radially symmetric solution of $-\Delta\psi = \psi$. Furthermore, there is $r_0 > 0$ such that v is strictly decreasing in $[0, r_0)$ and $v'(r_0) = 0, v(r_0) =: \kappa < 0$. For $R > 0$ the function φ_R defined by

$$\varphi_R(x) = \varphi(\frac{x}{R}) \quad \text{where} \quad \varphi(x) = (v(x) - \kappa) \cdot 1_{\{|x| \le r_0\}}$$

is positive in B_{Rr_0} with $\operatorname{supp}(\varphi_R) = \overline{B}_{Rr_0}$. By the choice of κ we have $\varphi \in C^{1,1}(\mathbb{R}^n)$ and Rademacher's theorem applied to $\partial_{x_i}\varphi$, $i = 1, \dots, n$ shows that $\varphi \in W_0^{2,\infty}(\mathbb{R}^n)$. Moreover, φ_R satisfies the differential equation

$$-\Delta\varphi_R + V(x)\varphi_R = \big((V(x) + \frac{1}{R^2})\varphi_R - \frac{|\kappa|}{R^2}\big) \cdot 1_{\{|x| \le Rr_0\}}$$

pointwise a.e. in $\mathbb{R}^n$. By Proposition 17 we may use φ_R as a test function in (1.2). Positivity of u and $-\Delta\varphi_R + V(x)\varphi_R \le (\|V\|_{L^\infty(\mathbb{R}^n)} + 1)\varphi_R$ almost everywhere for all $R \ge 1$ implies

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta\varphi_R + V(x)\varphi_R)u\,dx &\le (\|V\|_{L^\infty(\mathbb{R}^n)} + 1)\int_{B_{Rr_0}} \varphi_R u\,dx \\ &\le \frac{C_4}{2}\int_{B_{Rr_0}} u^p\varphi_R\,dx + c\int_{B_{Rr_0}} \varphi_R\,dx \\ &\le \frac{C_4}{2}\int_{B_{Rr_0}} u^p\varphi_R\,dx + cR^n\,.\end{aligned} \tag{3.7}$$

where C_4 is the constant from (1.3). Here we used that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $a \le \varepsilon a^p + C_\varepsilon$ for all $a > 0$. From the assumptions on g we get

$$\begin{aligned}\int_{\mathbb{R}^n} g(x,u)\varphi_R\,dx &\ge C_4\int_{B_{Rr_0}} u^p\varphi_R\,dx - C_3\int_{B_{Rr_0}} \varphi_R\,dx \\ &\ge C_4\int_{B_{Rr_0}} u^p\varphi_R\,dx - cR^n\,.\end{aligned} \tag{3.8}$$

Subtracting (3.8) from (3.7) we get

$$\frac{C_4}{2}\int_{B_{Rr_0}} u^p\varphi_R\,dx \le cR^n \quad \text{for all } R \ge 1. \tag{3.9}$$

For a fixed $\gamma \in (0,1)$ the function φ_R is uniformly bounded from below on $B_{\gamma Rr_0}$ so that (3.9) implies

$$\int_{B_{\gamma Rr_0}} u^p\,dx \le c_\gamma\int_{B_{Rr_0}} u^p\varphi_R\,dx \le c_\gamma\,R^n \quad \text{for all } R \ge 1. \tag{3.10}$$

Therefore we obtain the following inequality with $A_k := \{k\gamma r_0 \le |x| < (k+1)\gamma r_0\}$, $k \in \mathbb{N}_0$ and $\omega_0 > 0$:

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\sqrt{\omega_0}|x|} u(x)^p \, dx &= \sum_{k=0}^{\infty} \int_{A_k} e^{-\sqrt{\omega_0}|x|} u(x)^p \, dx \\
&\le c_\gamma \sum_{k=0}^{\infty} e^{-\gamma\sqrt{\omega_0} r_0 k} \int_{B_{\gamma r_0(k+1)}} |u(x)|^p \, dx \\
&\le c_\gamma \Big(1 + \sum_{k \ge \frac{1}{\gamma r_0} - 1}^{\infty} e^{-\gamma\sqrt{\omega_0} r_0 k} (\gamma r_0 (k+1))^n \Big) < \infty.
\end{aligned}$$

Hence, $u \in L^p(\mathbb{R}^n; \omega_0)$ for all $\omega_0 > 0$. $\square$

Proof of Theorem 3,(1): Let g satisfy (1.3) and let $u \ge 0$ be a distributional solution. Since $W := V - \Gamma |u|^{p-1} \in L^\infty(\mathbb{R}^n) + L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^n)$ and $\frac{p}{p-1} > \frac{n}{2}$ we find that W lies in the local Kato class K_n^{loc} (see [22],p.453) and thus Proposition 20 (applied to compact subsets of $\mathbb{R}^n$) gives $u \in L^\infty_{loc}(\mathbb{R}^n)$. From Proposition 18 we get $u \in L^p(\mathbb{R}^n; \omega)$ for all $\omega > 0$ so that Proposition 16 implies $u = T_\omega(g_\omega)$ where $g_\omega \in L^\infty_{loc}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; \omega)$ for all $\omega > 0$. Since $(k, q, r) = (1, \infty, \infty)$ is admissible for Proposition 14 we obtain $u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$, in particular

$$\int_{\mathbb{R}^n} \nabla u \nabla \varphi + \omega u \varphi \, dx = \int_{\mathbb{R}^n} u(-\Delta\varphi + \omega\varphi) \, dx = \int_{\mathbb{R}^n} g_\omega \varphi \, dx,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ so that u is a weak solution of the uniformly elliptic PDE (3.1) . From $g_\omega \in L^\infty_{loc}(\mathbb{R}^n)$ we obtain $u \in W^{2,q}_{loc}(\mathbb{R}^n)$ for all $q \in [1, \infty)$ by Caldéron-Zygmund estimates (cf. Chapter 9 in Gilbarg, Trudinger [6]). The same reasoning as in part (2) shows that u must be a strong solution in $\mathbb{R}^n$. $\square$

4. Appendix A

In the proof of Proposition 5 we use the following auxiliary lemma.

Lemma 19. *Let $0 < c_0 < 1$ and $\rho \ge 1$ be given. Then for all $p > 1$ there exists a radially symmetric positive function $u_2 \in C^\infty(\mathbb{R}^n \setminus B_\rho)$ such that*

$$\begin{aligned}
-\Delta u_2 + u_2 &= u_2^p \quad \text{in } \mathbb{R}^n \setminus B_\rho \\
u_2(x) &= c_0 \quad \text{for } |x| = \rho \qquad (4.1)\\
u_2(x) &\to 0 \quad \text{exponentially as } |x| \to \infty.
\end{aligned}$$

Moreover the following inclusion holds

$$0 < v(|x|) \le u_2(x) \le c_0 \, e^{-\sqrt{1-c_0^{p-1}}(|x|-\rho)} \quad \text{for all } |x| \ge \rho$$

where $v(r) = \kappa r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(r)$. Here $K_{\frac{n-2}{2}}$ denotes the modified Bessel function of second kind and $\kappa > 0$ is chosen such that $v(\rho) = c_0$.

Proof. We first use the method of sub- and supersolutions to find a solution $w_{2,R}$ of the following auxiliary elliptic ODE boundary value problem

$$\begin{aligned} -w_{2,R}'' - \frac{n-1}{r} w_{2,R}' + w_{2,R} &= w_{2,R}^p \qquad \text{in } (\rho, R), \\ w_{2,R}(\rho) = c_0, \quad w_{2,R}(R) &= v(R) \end{aligned} \tag{4.2}$$

for any given $R > \rho$. As a supersolution of (4.2) we may take the constant function c_0 since $c_0 \geq c_0^p$ and $c_0 = v(\rho) > v(R)$ using the fact that v is strictly decreasing. Since v is positive and satisfies the boundary conditions as well as

$$-v''(r) - \frac{n-1}{r} v'(r) + v(r) = 0 \qquad \text{in } (\rho, R)$$

we may choose v as a subsolution. Hence the method of sub- and supersolutions (cf. [30], §16) applies and produces a classical solution $w_{2,R}$ of (4.2) with the additional property

$$0 < v(r) \leq w_{2,R}(r) \leq c_0 < 1 \qquad \text{for } r > \rho. \tag{4.3}$$

The function $w_{2,R}$ cannot attain a local maximum at any $r^* \in (\rho, R)$ since in this case we would have $0 \leq -w_{2,R}''(r^*) = w_{2,R}(r^*)(w_{2,R}(r^*)^{p-1} - 1)$ contradicting (4.3). This implies that $w_{2,R}$ is decreasing since otherwise there would be $\rho \leq r_1 < r_2 < R$ such that $w_{2,R}(r_1) < w_{2,R}(r_2)$. Using that there is no interior local maximum this would lead to $w_{2,R}(r_1) < w_{2,R}(r_2) \leq w_{2,R}(R) = v(R)$ in contradiction to $w_{2,R}(r_1) \geq v(r_1) > v(R)$ by (4.3) and strict monotonicity of v.

Since $w_{2,R}$ is decreasing we have $w_{2,R}' \leq 0$ and from (4.2) and $w_{2,R} < 1$ we get $w_{2,R}'' > 0$, hence

$$0 \geq w_{2,R}'(r) \geq w_{2,R}'(\rho) \geq v'(\rho) \quad \text{ for all } r \in [\rho, R]. \tag{4.4}$$

From (4.2), (4.3) and (4.4) it follows that for all $R_0 > \rho$ the families $(w_{2,R}')_{R>R_0}$, $(w_{2,R}'')_{R>R_0}$ are uniformly bounded with respect to R. By the Arzelà-Ascoli theorem, there is a sequence (w_{2,R_j}) with $\lim_{j\to\infty} R_j = \infty$ which converges uniformly along with its first derivatives on every compact subset of $[\rho, \infty)$ to some $\tilde{u}_2 \in C^1([\rho,\infty))$ which satisfies the enclosure $0 < v \leq \tilde{u}_2 \leq c_0 < 1$. Writing

$$w_{2,R}(r) = c_0 + \frac{\rho}{2-n}\big((\frac{\rho}{r})^{n-2} - 1\big) w_{2,R}'(\rho) + \int_\rho^r \int_\rho^s (\frac{t}{s})^{n-1} [w_{2,R}(t) - w_{2,R}(t)^{2p-1}]\, dt\, ds$$

we obtain that $\tilde{u}_2 = \lim_{R\to\infty} w_{2,R}$ belongs to $C^2([\rho,\infty))$ and solves the initial value problem

$$-\tilde{u}_2'' - \frac{n-1}{r}\tilde{u}_2' + \tilde{u}_2 = \tilde{u}_2^p \quad \text{in } (\rho, \infty), \qquad \tilde{u}_2(\rho) = c_0 \tag{4.5}$$

in the classical sense. In particular, $u_2(x) := \tilde{u}_2(|x|)$ defines a radially symmetric classical solution of problem (4.1) on $\mathbb{R}^n \setminus B_\rho$. It remains to show that $\tilde{u}_2$ decays exponentially at infinity.

To this end we test (4.5) with functions $\varphi_k(r) := \varphi(r-k)$ for $k > 0$ and $\varphi \in C_0^\infty(\rho, \infty)$ arbitrary. Since $\tilde{u}_2 \in C^2([\rho,\infty))$ is a decreasing function it has a limit $\tilde{u}_{2,\infty} := \lim_{r\to\infty} \tilde{u}_2(r)$

which satisfies $0 \leq \tilde{u}_{2,\infty} < c_0 < 1$. Therefore the dominated convergence theorem implies

$$\begin{aligned}
0 &= \lim_{k\to\infty} \int_\rho^\infty \tilde{u}_2(r)\big(-\varphi_k''(r) - \frac{n-1}{r}\varphi_k'(r) + \varphi_k(r) - \tilde{u}_2(r)^{p-1}\varphi_k(r)\big)\,dr \\
&= \lim_{k\to\infty} \int_\rho^\infty \tilde{u}_2(r+k)\big(-\varphi''(r) - \frac{n-1}{r+k}\varphi'(r) + \varphi(r) - \tilde{u}_2(r+k)^{p-1}\varphi(r)\big)\,dr \\
&= \int_\rho^\infty \tilde{u}_{2,\infty}\big(-\varphi''(r) + \varphi(r) - \tilde{u}_{2,\infty}^{p-1}\varphi(r)\big)\,dr \\
&= \tilde{u}_{2,\infty}(1-\tilde{u}_{2,\infty}^{p-1}) \int_\rho^\infty \varphi(r)\,dr
\end{aligned}$$

and thus, φ being an arbitrary testfunction, we see that necessarily $\tilde{u}_{2,\infty} = 0$.

Finally we to show $\tilde{u}_2 \leq z$ where $z(r) := c_0 e^{-\sqrt{1-c_0^{p-1}}(r-\rho)}$. From $z''(r) = (1-c_0^{p-1})z$, $z(\rho) = c_0$ and $0 < \tilde{u}_2 \leq c_0, \tilde{u}_2' \leq 0$ we get

$$\begin{aligned}
(\tilde{u}_2 - z)''(r) &= -\frac{n-1}{r}\tilde{u}_2'(r) + \tilde{u}_2(r)(1-\tilde{u}_2(r)^{p-1}) - (1-c_0^{p-1})z(r) \\
&\geq (1-c_0^{p-1})(\tilde{u}_2 - z)(r) \qquad \text{for all } r \geq \rho.
\end{aligned}$$

which proves that $\tilde{u}_2 - z$ cannot have any positive interior local maximum. Hence,

$$(\tilde{u}_2 - z)(r) \leq \max\{0, (\tilde{u}_2 - z)(\rho), (\tilde{u}_2 - z)(\infty)\} = 0 \qquad \text{for all } r \geq \rho$$

and the result follows. □

Proof of Proposition 5: Let $n \geq 3$, choose ρ such that the inequalities

$$\rho \geq 1, \qquad \rho \geq \sqrt{\frac{4}{3}} \cdot \max\{c_{n,q}^{\frac{q-1}{2}} : \frac{n}{n-2} \leq q \leq \frac{n+2}{n-2}\}$$

hold true where $c_{n,p}$ is given by (2.1). Then, given any $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$ the choice $c_0 := c_{n,p}\rho^{-\frac{2}{p-1}}$ implies $0 < c_0 \leq c_{n,p}$ and $c_0^{p-1} \leq \frac{3}{4}$.

Let now u_2 be given by Lemma 19, $u_1(x) := c_{n,p}|x|^{-\frac{2}{p-1}}$. Then the function u_0 defined in (2.4) is positive radially symmetric and satisfies (i),(ii) by the choice of u_1, u_2. Moreover, $u_0 \in C(\mathbb{R}^n\backslash\{0\})$ implies $u_0 \in H^1(\mathbb{R}^n\backslash B_\delta)$ for all $\delta > 0$ and $u_1 \in C^2(\overline{B}_\rho\backslash\{0\}), u_2 \in C^2(\mathbb{R}^n\backslash B_\rho)$ gives (iii). Property (iv) follows from the definition of u_1. The explicit formula for u_1 and the enclosure of u_2 given by Lemma 19 yield

$$|\partial_\nu^+ u_0(x)| = |\partial_\nu u_1(x)| \leq c\, c_{n,p}, \qquad |\partial_\nu^- u_0(x)| = |\partial_\nu u_2(x)| \leq c\, c_{n,p} \quad (x \in \partial B_\rho)$$

and we obtain (v). By the choice of ρ we have $c_0^{p-1} \leq \frac{3}{4}$ so that Lemma 19 gives the upper bound for $u_2(x) \leq c_0\, e^{-\frac{|x|-\rho}{2}}$ which shows (vi) and finishes the proof of Proposition 5. □

5. Appendix B

The following proposition sums up two results from [22].

Proposition 20. *Let* $\Omega = \mathbb{R}^n \backslash B_R$ *for some* $R \geq 0$. *Let* $W \in K_n(\Omega)$ *and assume* $-\Delta u + Wu = 0$ *in* Ω *in the distributional sense where* $u, Wu \in L^1_{loc}(\Omega)$. *Then* u *equals almost everywhere a continuous function in* Ω. *If in addition* $u \in L^q(\Omega)$ *for some* $q \in [1, \infty)$ *then* $u(x) \to 0$ *as* $|x| \to \infty$.

Proof. Continuity of u follows from [22], Theorem C.1.1. Moreover [22], Theorem C.1.2. implies that for almost all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > 1$ we have

$$|u(x)| \leq C(\|W_-\|_{K_n(B_1(x))}) \int_{B_1(x)} |u(y)|\, dy \leq C(\|W_-\|_{K_n(\Omega)}) \int_{B_1(x)} |u(y)|\, dy.$$

Now if $u \in L^p(\Omega)$ we have $\lim_{|x|\to\infty} \int_{B_1(x)} |u(y)|^p\, dy = 0$ and thus Hölder's inequality implies $\lim_{|x|\to\infty} \int_{B_1(x)} |u(y)|\, dy = 0$. Hence the result. $\square$

Proposition 21. *Let* $\Omega = \mathbb{R}^n \setminus B_R$ *for some* $R \geq 0$. *Let* $W \in L^s(\Omega) + L^\infty(\Omega)$ *for some* $s > \frac{n}{2}$, *and assume* $0 < \Sigma := \inf \sigma_{ess}(-\Delta + W(x))$. *If* $u \in H^1_{loc}(\Omega) \cap L^q(\Omega)$ *for some* $q \in [2, \frac{2n}{(n-2)_+})$ *is a weak solution of* $-\Delta u + Wu = 0$ *in* Ω *then for all* $\mu \in (0, \sqrt{\Sigma})$ *there is a constant* $C_\mu > 0$ *such that*

$$|u(x)| \leq C_\mu e^{-\mu|x|} \qquad \text{for all } x \in \Omega \text{ with } \operatorname{dist}(x, \partial\Omega) > 1.$$

Proof. 1st step: Proof of exponential integrability

Let $\mu \in (0, \sqrt{\Sigma})$ be arbitrary and let $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi|_{B_1} \equiv 0$ and $\chi|_{B_2^c} \equiv 1$. Let $\chi_s(x) = \chi(s^{-1}x)$ for $x \in \mathbb{R}^n$ and $s > 0$. For $\rho > r > R$ we define the function

$$\chi_{r,\rho} := \chi_r \cdot (1 - \chi_\rho).$$

Notice that the support of $\chi_{r,\rho}$ is contained in the annulus $\overline{B}_{2\rho} \setminus B_r$ and $\chi_{r,\rho} \equiv \chi_r$ on $\overline{B}_\rho$. For $\sigma > 0$ we define

$$\varphi = \xi^2 u \quad \text{ where } \quad \xi(x) = \chi_{r,\rho}(x) e^{\frac{\mu|x|}{1+\sigma|x|}}.$$

Since $u \in H^1_{loc}(\Omega)$ is a weak solution of (1.1) in Ω and $\operatorname{supp}(\chi_{r,\rho}) \subset \overline{B}_{2\rho} \setminus B_r$ we have $\varphi \in H^1_0(\Omega)$ and

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla u \nabla \varphi + W u \varphi\, dx \\ &= \int_{\mathbb{R}^n} |\nabla(\xi u)|^2 + W|\xi u|^2 - |\nabla \xi|^2 |u|^2\, dx. \end{aligned}$$

Now fix a $\delta \in (0, \frac{1}{2}(\Sigma - \mu^2))$. From $|\nabla \xi| \leq e^{\frac{\mu|x|}{1+\sigma|x|}}(|\nabla \chi_{r,\rho}| + \mu|\chi_{r,\rho}|)$ we infer

$$|\nabla \xi|^2 \leq (\mu^2 + \delta)|\chi_{r,\rho}|^2 e^{\frac{2\mu|x|}{1+\sigma|x|}} + (1 + \mu^2\delta^{-1})|\nabla \chi_{r,\rho}|^2 e^{\frac{2\mu|x|}{1+\sigma|x|}}.$$

Hence,

$$0 \geq \int_{\mathbb{R}^n} |\nabla(\xi u)|^2 + W|\xi u|^2\, dx \tag{5.1}$$

$$-(\mu^2+\delta)\int_{\mathbb{R}^n}|\chi_{r,\rho}|^2|u|^2e^{\frac{2\mu|x|}{1+\sigma|x|}}\,dx-(1+\mu^2\delta^{-1})\int_{\mathbb{R}^n}|\nabla\chi_{r,\rho}|^2|u|^2e^{\frac{2\mu|x|}{1+\sigma|x|}}\,dx.$$

In view of $\inf\sigma_{ess}(W)\geq\Sigma$ and Persson's Theorem (cf. [9], Theorem 14.11.) we may choose $r>0$ so large that for all $\rho>r,\sigma>0$ the following inequality holds

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla(\xi u)|^2+W|\xi u|^2 &\geq(\Sigma-\delta)\int_{\mathbb{R}^n}|\xi u|^2\,dx \\ &=(\Sigma-\delta)\int_{\mathbb{R}^n}|\chi_{r,\rho}|^2|u|^2e^{\frac{2\mu|x|}{1+\sigma|x|}}\,dx.\end{aligned}\tag{5.2}$$

From (5.1) and (5.2) we get for all $\rho>r,\sigma>0$

$$\int_{\mathbb{R}^n}\chi_{r,\rho}^2|u|^2e^{\frac{2\mu|x|}{1+\sigma|x|}}\,dx\leq\frac{1+\mu^2\delta^{-1}}{\Sigma-\mu^2-2\delta}\int_{\mathbb{R}^n}|\nabla\chi_{r,\rho}|^2|u|^2e^{\frac{2\mu|x|}{1+\sigma|x|}}\,dx.\tag{5.3}$$

We want to take the limit $\rho\to\infty$. In the integral on the left-hand side of (5.3) this can be done by the monotone convergence theorem. If $q=2$ then the right-hand side of (5.3) can be treated by the dominated convergence theorem. In the case $2<q<\frac{2n}{(n-2)_+}$ notice that

$$\int_{\mathbb{R}^n}(|\nabla\chi_{r,\rho}|^2-|\nabla\chi_r|^2)^{\frac{q}{q-2}}\,dx=\int_{\{\rho\leq|x|\leq2\rho\}}|\nabla\chi_\rho|^{\frac{2q}{q-2}}\,dx\leq\|\nabla\chi\|_\infty\rho^{n-\frac{2q}{q-2}}\to0\quad\text{ as }\rho\to\infty.$$

Hence (5.3) holds with $\chi_{r,\rho}$ replaced by χ_r. Taking the limit $\sigma\to0$ we obtain

$$\int_{\mathbb{R}^n}\chi_r^2|u|^2e^{2\mu|x|}\,dx\leq\frac{1+\mu^2\delta^{-1}}{\Sigma-\mu^2-2\delta}\int_{\mathbb{R}^n}|\nabla\chi_r|^2|u|^2e^{2\mu|x|}\,dx<\infty.$$

The right-hand side is finite since $\nabla\chi_r$ has compact support. Hence, $\chi_r ue^{\mu|x|}\in L^2(\mathbb{R}^n)$ and thus $ue^{\mu|x|}\in L^2(\mathbb{R}^n\setminus B_{2r})$.

2nd step: Pointwise exponential decay

Since u is a weak solution of $-\Delta u+W(x)u=0$ in Ω the Harnack inequality (cf. [8], Theorem 4.1) implies that there is positive constant $C=C(\|W\|_{L^s(B_2(z))})$ such that

$$\|u\|_{L^\infty(B_1(z))}\leq C(\|W\|_{L^s(B_2(z))})\|u\|_{L^2(B_2(z))}.\tag{5.4}$$

for all $z\in\mathbb{R}^n$ with $|z|>2r+2$. Since $W\in L^s(\Omega)+L^\infty(\Omega)$ the constant C in (5.4) is w.l.o.g. independent of z. Hence, we get

$$\begin{aligned}\|ue^{\mu|\cdot|}\|_{L^\infty(B_1(z))}&\leq\|u\|_{L^\infty(B_1(z))}\|e^{\mu|\cdot|}\|_{L^\infty(B_1(z))}\\ &\leq C\|u\|_{L^2(B_2(z))}e^{\mu(|z|+1)}\\ &\leq C\|ue^{\mu|\cdot|}\|_{L^2(B_2(z))}e^{-\mu(|z|-2)}e^{\mu(|z|+1)}\\ &\leq Ce^{3\mu}\|ue^{\mu|\cdot|}\|_{L^2(\mathbb{R}^n\setminus B_r)}=:C_\mu.\end{aligned}$$

for $|z|>2r+2$ and thus $|u(x)|\leq C_\mu e^{-\mu|x|}$ for $|x|>2r+1$. Moreover by Proposition 20 u is bounded outside a neigbhourhood of $\partial\Omega$ and thus

$$|u(x)|\leq C_\mu e^{-\mu|x|}\qquad\text{for all } x\in\Omega \text{ with } \operatorname{dist}(x,\partial\Omega)>1.$$

□

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